

Lindbladians for controlled stochastic Hamiltonians

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Abstract

We construct Lindbladians associated with controlled stochastic Hamiltonians in weak coupling. This allows to determine the power spectrum of the noise from measurements of dephasing rates; to optimize the control and to test numerical algorithms that solve controlled stochastic Schrödinger equations. A few examples are worked out in detail.

1 The problem and the result

This article describes Lindbladians associated with controlled stochastic Hamiltonians in weak coupling. Controlled stochastic Hamiltonians arise in the context of “dynamical decoupling” and “coherent control” and are used to examine protocols for extending the coherence time of qubits [12, 9, 2].

Lindbladians in the weak coupling limit have been rigorously studied in [6, 13, 10, 4, 5, 18, 1, 16, 7] in the *time independent* setting. Recently, controls aimed at extending the coherence of qubits have been suggested in [12, 9] and periodically controlled Lindbladians have been studied in [2, 20]. However a careful derivation of the Lindbladians for the controlled stochastic evolutions and in particular Eq. (3.8) for general and in Eq. (1.10) for stationary control are new. This is also the case for the notion of effective control introduced in Appendix C.1 and some of the examples in section 4.

Consider the stochastic controlled Hamiltonian¹.

$$\underbrace{\left(\sum \xi_\alpha(t) H_\alpha\right)}_{\text{weak noises}} + \underbrace{H_c(t)}_{\text{control}}, \quad (1.1)$$

H_α are fixed Hermitian matrices representing independent and in general non-commuting sources of noise. ξ_α are stationary Gaussian random processes

$$\mathbb{E}(\xi_\alpha(t)) = 0, \quad \mathbb{E}(\xi_\alpha(t)\xi_\beta(u)) = J_{\alpha\beta}(|t-u|). \quad (1.2)$$

¹Controlled stochastic adiabatic evolutions are studied in [8].

with J rapidly decreasing on a time scale τ . We shall sometimes assume, w.l.o.g., that J is a diagonal matrix (this can be achieved by a redefinition of H_α). A spin in a magnetic field having fixed direction but noisy amplitude, often a good approximation [12, 9], is represented by a single term α . The case where the direction of the field is also stochastic is modeled by several α 's and gives rise to noise that is non-commutative (not been treated before.) H_c , a time-dependent (Hermitian) matrix, represents the control.

It is convenient to reformulate the problem in the interaction picture. Let

$$H_\xi^I(t) = \sum_\alpha \xi_\alpha(t) H_\alpha^I(t), \quad H_\alpha^I(t) = V^*(t) H_\alpha V(t) \quad (1.3)$$

where $V(t)$ is the unitary generated by the control $H_c(t)$,

$$H_c = i\dot{V}(t)V^*(t), \quad V(0) = \mathbb{1} \quad (1.4)$$

Weak coupling parameter in the present context is defined by

$$\varepsilon^2 = \tau \|\tilde{J}\| \|H_\alpha\|^2 \ll 1 \quad \tilde{J}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} J(t) dt \geq 0 \quad (1.5)$$

ε is the phase acquired by the wave function during one correlation time (in the absence of control). There are several ways to think about weak coupling: If we think of $\|\tilde{J}\|, \tau = O(1)$ then weak coupling means what its names suggests, namely, that the noise is weak in the sense that $\|H_\alpha\| = O(\varepsilon)$. An alternate approach, which is also insightful, is to take $\|\tilde{J}\|, \|H_\alpha\| = O(1)$ and then weak coupling means short correlation time $\tau = O(\varepsilon^2)$.

The noise affects the (average) state on the coarse grained time scale²

$$s = \varepsilon^2 t / \tau \quad (1.6)$$

Control problems are characterized by the rate of rotation of $H_\alpha^I(t)$. For example, when the control H_c is time independent, (constant control), $\omega = \|H_c\|$ while for periodic Bang-Bang, where $H_c(t)$ is a (periodic) sequence of delta pulses, ω_c is the frequency of the bangs. This gives rise to a second dimensionless parameter $\omega_c \tau$. Our analysis of the weak coupling limit holds independently of $\omega_c \tau$. Dynamical decoupling requires however $\omega_c \tau \gtrsim 1$ where the time scale of the control, $\delta t = O(1/\omega_c)$, is not resolved on the coarse grained time scale s .

By stationary controls we shall mean that $H_\alpha^I(t)$ has a finite number of Fourier coefficients. It is convenient to factor ε so that the Fourier coefficients are $\tilde{H}_\alpha(\omega)$ are

$$H_\alpha^I(t) = \varepsilon \sum_{\omega \in F} \tilde{H}_\alpha(\omega) e^{i\omega t}, \quad \tilde{H}_\alpha(\omega) = \tilde{H}_\alpha^*(-\omega) \quad (1.7)$$

F a finite set.

When $\varepsilon \ll 1$ we shall show that the evolution is governed by (complete) positivity preserving Lindbladian³

$$\frac{d\rho}{ds} = \mathcal{L}_\varepsilon \rho \quad (1.8)$$

²In contrast to evolution of the (unaveraged) state where time scales $O(\tau/\varepsilon)$ lead to effects of $O(1)$.

³This is related to the procedure of ‘‘adiabatic elimination’’[11].

Moreover, we shall show that, in the case of stationary control, \mathcal{L}_ε has a limit as $\varepsilon \rightarrow 0$ given by:

$$\mathcal{L} = \sum_{\alpha} \mathcal{L}_{\alpha}, \quad \mathcal{L}_{\alpha} = \mathcal{H}_{\alpha} - \mathcal{D}_{\alpha} \quad (1.9)$$

with

$$\begin{aligned} \mathcal{H}_{\alpha}\rho &= \frac{i\tau}{4} \sum_{\omega \in F} \tilde{K}_{\alpha}(\omega) [[\tilde{H}_{\alpha}(\omega), \tilde{H}_{\alpha}^*(\omega)], \rho], \\ \mathcal{D}_{\alpha}\rho &= \frac{\tau}{8} \sum_{\omega \in F} \tilde{J}_{\alpha}(\omega) [\tilde{H}_{\alpha}(\omega), [\tilde{H}_{\alpha}^*(\omega), \rho]] \end{aligned} \quad (1.10)$$

\tilde{J} denotes Fourier transform and K is the anti-symmetric partner of J :

$$K_{\alpha}(u) = i \operatorname{sgn}(u) J_{\alpha}(u), \quad (1.11)$$

Note that $\tilde{K}(\omega)$ is real and $\tilde{K}(0) = 0$. \mathcal{H}_{α} is a generator of unitary evolution since $[\tilde{H}_{\alpha}(\omega), \tilde{H}_{\alpha}^*(\omega)]$ is hermitian. Since $\tilde{J}(\omega) \geq 0$ $\mathcal{D}_{\alpha} \geq 0$ is a contraction, generating decoherence.

Remark 1.1. *The special form of \mathcal{L}_{α} reflects the fact that stochastic evolutions are unital: The fully mixed state $\rho \propto \mathbb{1}$ is stationary.*

Remark 1.2. *Eq. (1.10) can be used to determine \tilde{J} from the measured rates γ_{α} [3, 14, 19, 15]. See the examples in section 4. Moreover, it implies that the optimal measurement time is $t = O(\tau/\varepsilon)$. To see this observe that repeated measurements of a projection P in the state $\rho(t)$ generates a Poisson process with an average*

$$\operatorname{Tr}(\rho(t)P) = p(\gamma, t)$$

Given total allotted time T , an optimal estimator minimizes the standard deviation in γ . This fixes t to be the minimizer of the sensitivity

$$S = \frac{\sqrt{tp(1-p)}}{\left| \frac{dp}{d\gamma} \right|}$$

Eq. (1.10) determines $\rho(t)$ for $t \geq \tau/\varepsilon$. For a depolarizing qubit

$$\rho(t) = e^{-\gamma t} P + (1 - e^{-\gamma t}) \frac{1}{2} \mathbb{1}, \quad p(\gamma, t) = \frac{1 - e^{-\gamma t}}{2}$$

so S takes its minimum at the left edge of the interval, $t = O(\tau/\varepsilon)$.

2 Some exact results

The Hamiltonian H_ξ^I generates a stochastic unitary evolution U_ξ given by⁴

$$\begin{aligned} U_\xi(t) &= \left(e^{-i \int_0^t H_\xi^I(u) du} \right)_T \\ &= \sum_{n=0}^{\infty} (-i)^n \int_{0 \leq t_1 < t_2 < \dots < t_n \leq t} H_\xi(t_n) dt_n \dots H_\xi(t_1) dt_1 \end{aligned} \quad (2.1)$$

The time ordering, denoted by the subscript T in the first line is defined explicitly in the second. More crucial to us is the super-operator⁵ \mathcal{U}_ξ acting on states

$$\rho_0 \mapsto \rho_\xi(t) = \mathcal{U}_\xi \rho_0 = U_\xi(t) \rho_0 U_\xi^*(t) \quad (2.2)$$

The super-operator can be written similarly

$$\begin{aligned} \mathcal{U}_\xi(t) &= \left(e^{-i \int_0^t \mathcal{H}_\xi(s) ds} \right)_T \\ &= \sum_{n=0}^{\infty} (-i)^n \int_{0 \leq t_1 < t_2 < \dots < t_n \leq t} \mathcal{H}_\xi(t_n) dt_n \dots \mathcal{H}_\xi(t_1) dt_1 \end{aligned} \quad (2.3)$$

where the super-operators \mathcal{H} acts by the adjoint action

$$\mathcal{H}\rho = (ad[H])\rho \equiv [H, \rho] \quad (2.4)$$

Note that $ad[H_1]ad[H_2] \neq ad[H_1 H_2]$. Rather,

$$(ad[H_1]ad[H_2])(\rho) = (\mathcal{H}_1 \mathcal{H}_2)(\rho) = \mathcal{H}_1(\mathcal{H}_2 \rho) = [H_1, [H_2, \rho]] \quad (2.5)$$

We also need the fact that

$$ad[[A, B]] = [ad[A], ad[B]] \quad (2.6)$$

which follows from Jacobi's identity.

The key object of this study is the (stochastic) averaged evolution

$$\rho_0 \mapsto \mathbb{E}(\rho_\xi(t)) = (\mathcal{U}(t))\rho_0 \quad (2.7)$$

The super-operator $\mathcal{U}(t)$ is trace preserving, (completely) positivity preserving and unital (i.e. $\mathcal{U}\mathbb{1} = \mathbb{1}$), but, in general, not unitary or Markovian.

Recall that for Gaussian averages

$$\mathbb{E}(e^{i\phi}) = e^{-\mathbb{E}(\phi^2)/2} \quad (2.8)$$

⁴Since we are interested in the case $\tau > 0$ we can avoid Ito's calculus.

⁵We shall use script characters to denote super-operators.

It follows that for ξ a stationary Gaussian process,

$$\begin{aligned}\mathcal{U}(t) &= \left(\exp \left(-\frac{1}{2} \int_0^t du \int_0^t dv \mathcal{K}(u, v) \right) \right)_T \\ &= \left(\exp \left(-\int_0^t du \int_0^u dv \mathcal{K}(u, v) \right) \right)_T\end{aligned}\tag{2.9}$$

where

$$\mathcal{K}(u, v) = \mathbb{E} (\mathcal{H}_\xi^I(u) \mathcal{H}_\xi^I(v)) = \sum_{\alpha\beta} J_{\alpha\beta}(u-v) \mathcal{H}_\alpha^I(u) \mathcal{H}_\beta^I(v) \tag{2.10}$$

So far, no approximation has been made. However, the time ordering remains a major complication⁶. For its precise meaning one can either go back to Eq. (2.3), or alternatively, see the discussion and graphical representation in Appendix A. There is no issue with time ordering in two cases: when ξ is white noise and when the (interaction picture) Hamiltonian commute at different times. We examine these cases first.

2.1 White noise

White noise is the limit $\tau \rightarrow 0$ with $\tau J = O(1)$. By Eq. (1.5) this corresponds to $\varepsilon \propto \sqrt{\tau} \rightarrow 0$. Not surprisingly, the reduction of white noise to Lindblad evolution is exact. Since $J_{\alpha\beta}(t) = J_{\alpha\beta} \delta(t)$ we have

$$\mathcal{U}(t) = \left(\exp \left(\int_0^t du \mathcal{L}(u) \right) \right)_T \tag{2.11}$$

with

$$\mathcal{L}(t) = -\frac{1}{2} \sum_{\alpha\beta} J_{\alpha\beta} \mathcal{H}_\alpha^I(t) \mathcal{H}_\beta^I(t) \tag{2.12}$$

Since \mathcal{L} and \mathcal{H}^I have the same time argument t , we may use the definition of time-ordering in Eq. (2.3) with $\mathcal{H}_\xi \mapsto \mathcal{L}$, to conclude that \mathcal{L} is the generator of \mathcal{U} . The Lindbladian reduces to:

$$\mathcal{L}_t \rho = -\frac{1}{2} \sum_{\alpha\beta} J_{\alpha\beta} [H_\alpha^I(t), [H_\beta^I(t), \rho]] \tag{2.13}$$

The result is exact. Since $H_\alpha(t)$ are unitarily related for different t it follows that the family \mathcal{L}_t is unitarily related and hence isospectral. In particular, the instantaneous dephasing rates are independent of the control $V(t)$. This could be anticipated since to affect the dephasing rates, the control must be at least as fast as the noise correlations.

⁶ \mathcal{U} may be viewed as the grand canonical partition function of a 1-D quantum gas with short range interaction.

2.2 Commutative case

In general, it is difficult to extract a generator of the evolution from Eq. (2.9) because of the time ordering. In the commutative case this is not an issue and the generator of the evolution follows from Eq. (2.10). Let us denote

$$\mathcal{G}(t) = - \int_0^t du \mathcal{K}(t, u)_T = - \sum_{\alpha} \int_0^t du J_{\alpha\beta}(t - u) (\mathcal{H}_{\alpha}^I(t) \mathcal{H}_{\beta}^I(u))_T \quad (2.14)$$

We then have

$$\mathcal{U}(t) = \left(\exp \left(\int_0^t du \mathcal{G}(u) \right) \right)_T \quad (2.15)$$

Of course, in the commutative case the index T is redundant. Now although \mathcal{G} is an exact generator, it is not in general of Lindblad form. More precisely, it may fail to satisfy positivity at short times as the following example shows.

Example 2.1 (Commutative case). *The commutative case arises, for example, when the (interaction picture) noise has a stochastic amplitude but a fixed “direction”, i.e. when*

$$H_{\xi}^I(t) = \xi(t) H_0$$

Since $H_{\xi}^I(t)$ is a commuting family, Eq. (2.15) is exact and the generator of the evolution is

$$\mathcal{G}(t)\rho = -\frac{\gamma(t)}{2} \mathcal{H}_0 \mathcal{H}_0 \rho = -\frac{\gamma(t)}{2} [H_0, [H_0, \rho]], \quad (2.16)$$

The “dephasing rate” $\gamma(t)$ is given by

$$\gamma(t) = 2 \int_0^t du J(t - u) = 2 \int_0^t du J(u) \quad (2.17)$$

Although $\gamma(0) \geq 0$ for very short times (since $J(0) > 0$), $\gamma(t)$ may be negative for $t = O(\tau)$ ⁷ as in Fig. 1. In these cases $\mathcal{G}(t)$ does not generate a contraction for all times. This reflects the fact that the evolution is not strictly Markovian. At longer times, $t \gg \tau$, one always has $\gamma(t) > 0$ (since $\tilde{J}(0) \geq 0$).

Positivity is regained in the weak coupling limit. Here it is convenient to consider the limit in the sense of short correlation time so $\tau = \varepsilon^2$. We get from Eqs. (1.6, 1.8)

$$\mathcal{L} = \mathcal{G}, \quad (\tau = \varepsilon^2) \quad (2.18)$$

In the limit $\varepsilon \rightarrow 0$, we get for $s > 0$,

$$\mathcal{L}\rho = -\frac{\tilde{J}(0)}{2} [H_0, [H_0, \rho]], \quad (2.19)$$

with time independent positive dephasing rate:

$$0 < \tilde{J}(0) = \int_{-\infty}^{\infty} J(u) du = \lim_{\varepsilon \rightarrow 0} 2\tau \int_0^{s/\varepsilon^2} J(u\tau) du \quad (2.20)$$

⁷Take e.g. $J(\omega) \propto \delta(\omega - \omega_0) + \delta(\omega + \omega_0)$

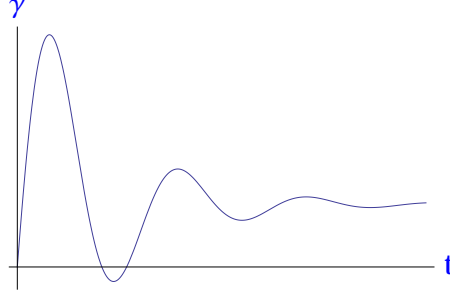


Figure 1: The figure shows $\gamma(t)$ for $J(t) = e^{-|t|} \cos 4t$ with $\gamma < 0$ near the first minimum.

3 Weak coupling

Our aim is to obtain an approximate generator that is valid for small nonzero ε . The first step is to show that \mathcal{G} defined in Eq. (2.14) remains an approximate generator in the noncommutative case. More precisely, moving the time ordering T in the exact formula Eq. (2.9) into the exponential, comes with the penalty:

$$\begin{aligned} \mathcal{U}(t) &= \left(\exp \left(-\frac{1}{2} \int_0^t du \int_0^t dv \mathcal{K}(u, v) \right) \right)_T \\ &= \exp \left(\int_0^t du \mathcal{G}(u) \right)_{T_u} + O \left(\frac{\varepsilon^4 t}{\tau} \right) \end{aligned} \quad (3.1)$$

Here T_u denotes time ordering with respect to the integration variable u . This differs from the usual time ordering defined with respect to the argument of the hamiltonian. The error term results from the inequivalence of the two types of ordering. It is proportional to t and hence is cumulative. It reflects the non-commutativity of the Hamiltonian at different times (there is no error in the commutative case as we have seen in section 2.2). In particular, on the coarse grained time scale $s = O(1) \Leftrightarrow t = O(\tau/\varepsilon^2)$ the error is $O(\varepsilon^2)$. We justify this estimate in Appendix A.

As we have seen (again in section 2.2) \mathcal{G} may not be a Lindbladian. Our next step is to show that within the framework of weak coupling, \mathcal{G} is close to a generator that is of the Lindbladian form up to an $O(\varepsilon^2)$ error.

To show this we introduce a useful representation of the noise ξ in terms of white noises W_α :

$$\xi_\alpha(t) = \int_{-\infty}^{\infty} du j_{\alpha\beta}(t-u) W_\beta(u) \quad (3.2)$$

(summation implied) where

$$\mathbb{E}(W_\alpha(t)) = 0, \quad \mathbb{E}(W_\alpha(t)W_\beta(u)) = \delta_{\alpha\beta}\delta(t-u) \quad (3.3)$$

There is freedom in defining j which allows us to assume, w.l.o.g., that its Fourier transform is non-negative, $\tilde{j}(\omega) \geq 0$. J is then the convolution of j with itself:

$$J_{\alpha\beta}(u-v) = \int_{-\infty}^{\infty} dw j_{\alpha\gamma}(u-w) j_{\beta\gamma}(v-w) \quad (3.4)$$

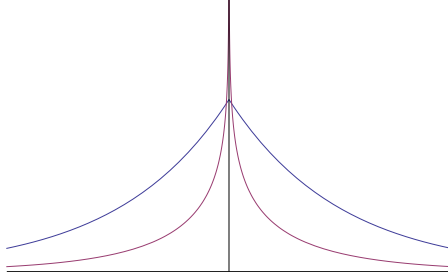


Figure 2: $J(t) = e^{-|t|}$ and the corresponding $j(t) = K_0(|t|)$, a Bessel function. j is narrower than J .

With this notation in place we first note the identity

$$\begin{aligned} \left(\int_0^t dv \int_0^t du \mathcal{K}(u, v) \right)_T &= \left(\int_0^t dv \int_0^t du J_{\alpha\beta}(u-v) \mathcal{H}_\alpha^I(u) \mathcal{H}_\beta^I(v) \right)_T \\ &= \sum_\gamma \int_{-\infty}^{\infty} dw \left(\int_0^t du j_{\alpha\gamma}(w-u) \mathcal{H}_\alpha^I(u) \right)_T^2 \end{aligned} \quad (3.5)$$

which follows from Eq. (3.4).

The second step is the claim that we can interchange the limits of the dw and du integration in Eq. (3.5) up to a small error, i.e.

$$\int_{-\infty}^{\infty} dw \left(\sum_\alpha \int_0^t du j_{\alpha\gamma}(w-u) \mathcal{H}_\alpha^I(u) \right)_T^2 = \int_0^t dw (\mathcal{D}_\gamma(w))_T^2 + O(\varepsilon^2) \quad (3.6)$$

where

$$\mathcal{D}_\gamma(w) = \frac{1}{2} \sum_\alpha \int_{-\infty}^{\infty} du j_{\alpha\gamma}(w-u) \mathcal{H}_\alpha^I(u) \quad (3.7)$$

This follows from the fact that $j(u)$ is localized near the origin on a time scale $O(\tau)$. It is clear that the main contribution to the integral comes from the region where both $w, u \in [0, t]$. The error corresponds to contributions where w, u are in an $O(\tau)$ neighborhood of the interval endpoints (see Fig. (3)). As this region has volume $O(\tau^3)$ and the integrand is $O(j^2 H^2)$ we conclude that the error is of magnitude $\tau^3 j^2 H^2 \sim \tau^2 J H^2 \sim \varepsilon^2$, whereas the first term is of order $t \tau^2 j^2 H^2 \sim t \tau J H^2 \sim t \varepsilon^2 / \tau \sim s$ which dominates the error. This proves Eq. (3.6) with an error uniform in time.

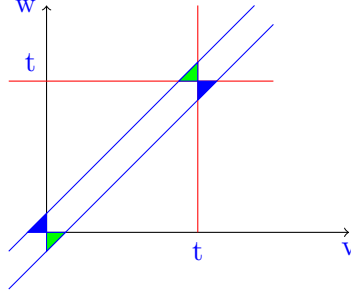


Figure 3: The figure illustrates the error terms in Eq. (3.6) due to the change of domain of integration in Eq. (3.5). The error terms are represented by the blue and green triangles. The width of the strip bounded by the blue lines is τ . Note that the figure is two dimensional while the actual domain of integration is three dimensional.

It follows that for ε small the (time-dependent) super-operator

$$\mathcal{G}_\varepsilon = -\frac{1}{2} \sum_{\alpha} (\mathcal{D}_\alpha(t)^2)_T \quad (3.8)$$

generates a CP map which is $O(\varepsilon^2)$ close to \mathcal{U} .

To rephrase \mathcal{G}_ε in terms of operators, rather than super-operators, use

$$2(\mathcal{H}(u)\mathcal{H}(v))_T = \{\mathcal{H}(u), \mathcal{H}(v)\} + \text{sgn}(u-v)[\mathcal{H}(u), \mathcal{H}(v)] \quad (3.9)$$

and the dictionary in Eq. (2.5,2.6) gives a time dependent generator

$$\begin{aligned} \mathcal{G}_\varepsilon \rho &= i[H^{ren}(t), \rho] - \frac{1}{2} \sum_{\alpha} [D_\alpha(t), [D_\alpha(t), \rho]] \\ H^{ren}(t) &= \frac{i}{4} \sum_{\alpha} \int j_\alpha(u) j_\alpha(v) \text{sgn}(u-v) [H_\alpha^I(t+u), H_\alpha^I(t+v)] du dv \end{aligned} \quad (3.10)$$

Since the operators $D_\alpha(t)$ and $H^{ren}(t)$ are self-adjoint \mathcal{G}_ε is a bona-fide time dependent generator of a CP map.

3.1 Coarse graining: The Lindbladian in the $\varepsilon \rightarrow 0$ limit

So far we kept ε small but finite and allowed arbitrary time dependence of $H^I(t)$. This gives the time dependent generator of the previous section. To properly define the limit $\varepsilon \rightarrow 0$, one should also specify the limiting behavior of the dimensionless parameter $\omega_c \tau$. If $\omega_c \tau \rightarrow 0$ then τ is the smallest time scale and ξ becomes effectively equivalent to white noise discussed in section 2.1. The interesting case and the one relevant to dynamic decoupling is when $\omega_c \tau \geq O(1)$ (where $\omega_c t = \omega_c \tau s / \varepsilon^2 \rightarrow \infty$). In this limit Eq. (3.10) reduces to Eq. (1.10). To see this note:

- Weak coupling may be interpreted as $J, \tau, \omega_c = O(1)$ while $\|H_\alpha\| = O(\varepsilon)$. Eq. (1.7) then implies that $\tilde{H}_\alpha(\omega) = O(1)$.
- The ansatz Eq. (1.7) says that the integral in Eq. (3.7) reduces to the sum:

$$\mathcal{D}_\alpha(t) = \frac{\varepsilon}{2} \sum_{\omega \in F} \tilde{j}(\omega) e^{i\omega t} \text{ad}(\tilde{H}_\alpha(\omega)) \quad (3.11)$$

- The limit $\varepsilon \rightarrow 0$ means that

$$\lim_{\varepsilon \rightarrow 0} e^{i\omega \tau s / \varepsilon^2} = \begin{cases} 0 & \omega \neq 0 \\ 1 & \omega = 0 \end{cases} \quad (3.12)$$

in the sense of distributions.

- The limiting Lindbladian generates the evolution on the time scale $s = \varepsilon^2 t / \tau$, it is related to \mathcal{G}_ε by $\mathcal{L} = \tau \varepsilon^{-2} \mathcal{G}_\varepsilon$.

It follows that for the second term in Eq. (3.10) we get

$$\frac{\tau}{2\varepsilon^2} \mathcal{D}_\alpha^2(t) \xrightarrow{\varepsilon \rightarrow 0} \frac{\tau}{8} \sum_{\omega \in F} \tilde{j}_\alpha(\omega) \tilde{j}_\alpha(-\omega) \text{ad}(\tilde{H}_\alpha(\omega)) \text{ad}(\tilde{H}_\alpha(-\omega))$$

which is \mathcal{D}_α of Eq. (1.10). Similarly, for the first term in Eq. (3.10) we get

$$i \frac{\tau}{\varepsilon^2} \text{ad}(H^{\text{ren}}) \xrightarrow{\varepsilon \rightarrow 0} \frac{i\tau}{4} \sum_{\alpha\omega} \int j_\alpha(u) j_\alpha(v) \text{sgn}(v-u) e^{i\omega(u-v)} \text{ad}([\tilde{H}_\alpha(\omega), \tilde{H}_\alpha^*(\omega)]) dudv$$

The u, v integration can be carried out explicitly to give

$$\int j_\alpha(u) j_\alpha(v) \text{sgn}(v-u) e^{i\omega(u-v)} dudv = \int J_\alpha(u) \text{sgn}(u) e^{i\omega u} = \tilde{K}_\alpha(\omega)$$

4 Examples

4.1 No control

Consider the (commutative) stochastic Hamiltonian for spin \mathbf{S}

$$H = \xi S_z \quad (4.1)$$

Eq. (2.19) gives the dephasing Lindbladian

$$\mathcal{L}\rho = -\frac{\gamma}{2} [S_z, [S_z, \rho]], \quad \gamma = \tilde{J}(0) \quad (4.2)$$

with a single rate parameter γ . The coherences can be computed using the spectral properties of the super-operator of angular momentum given in Appendix B

$$\text{Spectrum}\left((\text{ad}(S_z))^2\right) = \{m^2 \mid m \in \{0, \dots, 2S\}\} \quad (4.3)$$

The coherence decreases quadratically with the polarization m : $[S_z, [S_z, |m_1\rangle \langle m_2|]] = |m_1\rangle \langle m_2| (m_1 - m_2)^2$.

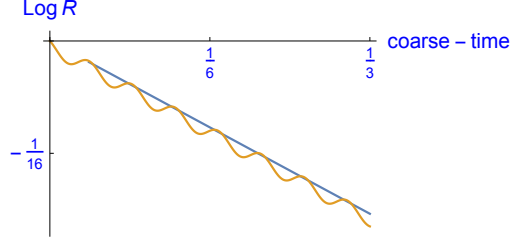


Figure 4: For a qubit state $\rho = (\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma})/2$ the figure shows $2 \log |\mathbf{r}|$ as a function of coarse grained time s for constant control. The parameters are the same as in Fig. 7, i.e. $\varepsilon = 0.15$ and $\omega_c \tau = \pi/2$. The figure compares the average Lindblad evolution on the coarse grained time scale, Eq. (1.10), and the time dependent Lindblad evolution, Eq. (3.10). Note the different scales in the two figures. The oscillations are tiny.

4.2 Bang-Bang

“Bang-bang” makes γ smaller and improves the coherence: A sequence of rapid π rotations about an axis perpendicular to the magnetic field self-average the noise. The π rotations are given by the unitary:

$$V(t) = \begin{cases} e^{i\pi S_x} & \omega t \bmod 2\pi \in (-\pi, 0) \\ \mathbb{1} & \omega t \bmod 2\pi \in (0, \pi) \end{cases}$$

The controlled stochastic Hamiltonian corresponding to Eq. (4.1) then takes the form:

$$H_\xi^I(t) = \xi(t) S_z w(\omega t) \quad (4.4)$$

where w is the square wave

$$w(t) = \begin{cases} 1 & \omega t \bmod 2\pi \in (-\pi, 0) \\ -1 & \omega t \bmod 2\pi \in (0, \pi) \end{cases}$$

Using the Fourier expansion

$$W(t) = -\frac{2i}{\pi} \sum_n \frac{e^{i(2n+1)t}}{2n+1}$$

and Eq. (1.10), we obtain the functional form of the dephasing Lindbladian of Eq. (4.2) but with a renormalized⁸ $\gamma \mapsto \gamma_b(\omega)$:

$$\gamma_b(\omega) = \frac{8}{\pi^2} \sum_{n \geq 0} \frac{\tilde{J}((2n+1)\omega)}{(2n+1)^2} \quad (4.5)$$

Since, $\tilde{J}(\omega) \rightarrow 0$ as $\omega \rightarrow \infty$, in the limit $\gamma_b(\omega) \rightarrow 0$, and there is no loss of coherence.

⁸For a monotonically decreasing $\tilde{J}(\omega)$ one has $\gamma_b(\omega) < \gamma_b(0) = \gamma$.

4.3 Constant control

Consider the stochastic Hamiltonian with time-independent control

$$H = \underbrace{\omega_c S_z}_{\text{control}} + \xi S_x \quad (4.6)$$

The control is effective in the sense of Appendix C.1. In the interaction picture the stochastic Hamiltonian has the form

$$H_\xi^I(t) = \xi(t) (S_x \cos \omega_c t + S_y \sin \omega_c t) \quad (4.7)$$

The frequency set F in Eq. (1.7) has two elements, $F = \{\pm\omega_c\}$ and

$$\tilde{H}(\pm\omega_c) = \frac{1}{2}(S_x \pm iS_y)$$

we find, from Eqs. (1.10) the Lindbladian

$$\mathcal{L}\rho = -\frac{i\tilde{K}(\omega_c)}{8}[S_z, \rho] - \frac{\tilde{J}(\omega_c)}{2} \sum_{j \in x, y} [S_j, [S_j, \rho]] \quad (4.8)$$

The two terms in \mathcal{L} commute. This follows from the fact that $J_i \equiv \text{ad}(S_i)$, $i = x, y, z$ give an $SU(2)$ representation. As in any such representation $J_x^2 + J_y^2$ is invariant under rotation around the z -axis, one has $[J_z, J_x^2 + J_y^2] = 0$. This may also be verified directly by calculating the commutators.

It follows that the first term in \mathcal{L} determines the imaginary part of the eigenvalues while the second term determines the real part. The coherence is then determined by the spectrum of

$$\text{spectrum}\left(\sum_{j=x, y} \text{ad}(S_j)\text{ad}(S_j)\right) = \begin{cases} \{0, 1^{(2)}, 2\} & S = 1/2 \\ \{0, 1^{(2)}, 2^{(3)}, 5^{(2)}, 6\} & S = 1 \end{cases}$$

and the index denotes multiplicities. In the case of general S the spectrum is $\{j(j+1) - m^2 \mid |m| \leq j \leq 2S\}$ as computed in Appendix B. In particular using Schur's lemma implies that 0 is always a simple eigenvalue. It follows that the Lindbladian is depolarizing: The unique equilibrium state is the fully mixed state.

4.4 Non-commutative noise

The simplest case of non-commutative noise is “planar” noise

$$\sum_{\alpha=x, y} \xi_\alpha S_\alpha$$

Eq. (1.10) gives the depolarizing Lindbladian

$$\mathcal{L}\rho = -\frac{1}{2} \sum_{j \in x, y} \gamma_j [S_j, [S_j, \rho]], \quad \gamma_j = \tilde{J}_j(0) \quad (4.9)$$

$H_c = \omega S_z$ is an effective control. Moreover, the results of sections 4.2, 4.3 carry over to this case, mutatis mutandis: Bang-Bang leads to Eq. (4.9) with $\gamma \mapsto \gamma_b$ as in Eq. (4.5). Constant control leads to an equation similar to Eq. (4.8) up to an extra factor of 2.

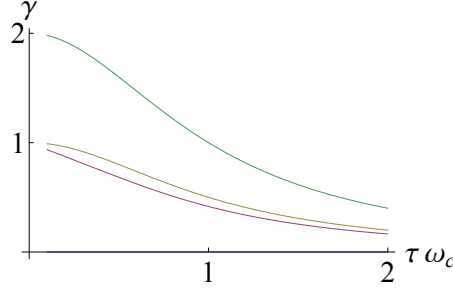


Figure 5: The dephasing rate γ for $\tilde{J}(\omega_c) = (1 + (\omega_c \tau)^2)^{-1}$ for a spin qubit in a stochastic magnetic field with a fixed direction as function of the rate of control $\tau \omega_c$. The lowest line corresponds to Bang-bang of Eq. (4.5). The two upper lines show the two nonzero eigenvalue of constant control. (Note that setting $\omega_c = 0$ in the graph is not meaningful since Eq. (3.12) fails.)

4.5 Isotropic noise

Isotropic noise is represented by the Hamiltonian

$$\sum_{\alpha=1}^3 \xi_{\alpha} S_{\alpha}, \quad J_1 = J_2 = J_3$$

leading to the isotropic depolarizing Lindbaldian

$$\mathcal{L}\rho = -\frac{\tilde{J}(0)}{2} \sum_{j=1}^3 [S_j [S_j, \rho]],$$

For $S = 1/2$ constant control is not effective⁹. One can, however, find an effective Bang-Bang.¹⁰

The simplest version of bang-bang about all three axes is associated with the unitary V

$$V(t) = \begin{cases} \sigma_1 & \omega t \bmod 2\pi \in [0, \pi/2] \\ \sigma_2 & \omega t \bmod 2\pi \in [\pi/2, \pi] \\ \sigma_3 & \omega t \bmod 2\pi \in [\pi, 3\pi/2] \\ \mathbb{1} & \omega t \bmod 2\pi \in [3\pi/2, 2\pi] \end{cases} \quad (4.10)$$

This control self-averages the Hamiltonian in the interaction picture to zero but leads to a non-isotropic Lindblad equation (with $\gamma_1 = \gamma_3 \neq \gamma_2$). In order to retain isotropy,

⁹For $S = 1$ a possible effective constant control is $H_c = \sum \alpha_i S_i^2$.

¹⁰The generalization to arbitrary spin is quite simple and only requires replacing the σ_k matrices in Eqs. (4.10, 4.11) by the appropriate rotation operator $R_k = \exp(i\pi S_k)$.

we choose a somewhat more complicated $V(t)$ corresponding to dividing $[0, 2\pi]$ into 12 equal parts¹¹. We demand $V(t) = V_j$ for $\omega t \bmod 2\pi \in [2\pi j/12, 2\pi(j+1)/12]$ where

$$\{V_j\}_{j=1}^{12} = \{\sigma_1, \sigma_2, \sigma_3, \mathbb{1}, \sigma_2, \sigma_3, \sigma_1, \mathbb{1}, \sigma_3, \sigma_1, \sigma_2, \mathbb{1}\} \quad (4.11)$$

This gives the stochastic Hamiltonian

$$H_\xi = \sum \xi_\alpha S_\alpha w_\alpha(\omega t) \quad (4.12)$$

where $w_1(t) = w_2(t+2\pi/3) = w_3(t-2\pi/3) = \pm 1$ takes on $[2\pi j/12, 2\pi(j+1)/12] \bmod 2\pi$ the values

$$w_1(t) \Leftrightarrow \{+1, -1, -1, +1, -1, -1, +1, +1, -1, +1, -1, +1\}_j \quad (4.13)$$

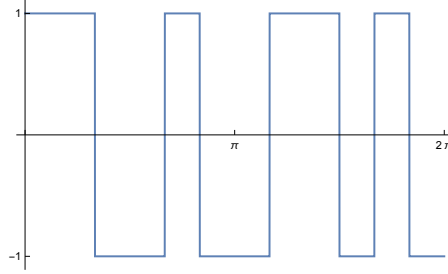


Figure 6: The Bang-Bang control corresponding to the square wave $w_1(\omega t)$ as function of time. It self-averages the noise Hamiltonian to zero while preserving the isotropy of the Lindbladian.

By symmetry considerations, $\mathcal{H}_\alpha = 0$ (since $[\tilde{H}_\alpha(\omega), H_\alpha^*(\omega)] \propto [S_\alpha, S_\alpha] = 0$). and the depolarizing Lindbladian has renormalized rates:

$$\mathcal{L}\rho = -\frac{\gamma(\omega)}{2} \sum_{j=1}^3 [\sigma_j [\sigma_j, \rho]],$$

$\gamma(\omega)$ is a more complicated version of Eq. (4.5)

$$\gamma(\omega) = \frac{8}{\pi^2} \sum_{n \neq 0} \frac{\tilde{J}(\omega n)}{n^2} \sin^4\left(\frac{n\pi}{12}\right) p(n) \quad (4.14)$$

$$p(n) = 5 + 4 \cos\left(\frac{n\pi}{6}\right) + 2 \cos\left(\frac{4n\pi}{3}\right) + (-1)^n \left(1 + 4 \cos\left(\frac{n\pi}{2}\right) + 2 \cos\left(\frac{2n\pi}{3}\right)\right)$$

¹¹We thank Ori Hirschberg for this suggestion.

4.6 Stochastic Harmonic oscillator

The stochastic harmonic oscillator provides a good model for trapped atoms, mechanical oscillators and trapped ions [17]. Since $\mathbb{1}$ is not a state in an infinite dimensional Hilbert space, the Lindbladian associated with stochastic evolution may have no stationary state.

There are various types of noises one may consider. The first is

$$H_\xi = \underbrace{\frac{1}{2}\omega_c(p^2 + x^2)}_{H_0} + \xi_p p + \xi_x x$$

with ξ_x and ξ_p Gaussian (possibly correlated) processes. This is known as ‘linear noise’ since it does not affect the frequency of the oscillator.

The interaction Hamiltonian is

$$H_\xi^I = \xi_x (x \cos \omega_c t + p \sin \omega_c t) + \xi_p (p \cos \omega_c t - x \sin \omega_c t)$$

H_0 is an effective control since H_ξ^I has vanishing time average. Observe that $ad(x)$ and $ad(p)$ commute since

$$[ad(x), ad(p)] = ad([x, p]) = i ad(\mathbb{1}) = 0 \quad (4.15)$$

It follows from Eq. (1.10) that the Lindbladian is real (has no Hamiltonian piece) and has the form

$$-2\mathcal{L} = \Gamma_x ad(x)ad(x) + \Gamma_p ad(p)ad(p) + 2\Gamma_{xp}\{ad(x), ad(p)\}$$

with matrix $\Gamma \propto \tilde{J}(\omega_c)$ at the oscillator frequency. Since $ad(x)$ and $ad(p)$ commute, and $spect(ad(x)) = spect(ad(p)) = (-\infty, \infty)$ and Γ is a positive matrix, we have

$$spect(\mathcal{L}) = \{-(\eta, \Gamma\eta) \mid \eta \in \mathbb{R}^2\} = (-\infty, 0]$$

0 is in the spectrum but is not associated with an eigenvalue: There is no stationary equilibrium state.

In the case of noise in the frequency of the harmonic oscillator the Hamiltonian is:

$$H_\xi = \frac{1}{2}((\omega_c + 2\xi_p)p^2 + (\omega_c + 2\xi_x)x^2) = \underbrace{\frac{1}{2}\omega_c(p^2 + x^2)}_{H_0} + \xi_p p^2 + \xi_x x^2$$

In the interaction picture one has

$$H_\xi^I = \xi_p (p \cos \omega_c t - x \sin \omega_c t)^2 + \xi_x (x \cos \omega_c t + p \sin \omega_c t)^2$$

Hence

$$\tilde{H}_\alpha(0) = [\tilde{H}_\alpha(2\omega_c), \tilde{H}_\alpha(-2\omega_c)] = \frac{1}{2}(p^2 + x^2), \quad \tilde{H}_\alpha(\pm 2\omega_c) = \frac{1}{4}(-1)^\alpha (p \pm ix)^2$$

and the Lindbladian is :

$$\mathcal{L} = \underbrace{K_2 \text{ad } \tilde{H}(0)}_{\text{unitary}} + \underbrace{\Gamma_0 (\text{ad } \tilde{H}(0))^2}_{\text{dephasing}} + \underbrace{\Gamma_2 \left((\text{ad } \tilde{H}(2\omega_c))^2 + (\text{ad } \tilde{H}(-2\omega_c))^2 \right)}_{\text{parametric drive}} \quad (4.16)$$

where $K_2 \propto \tilde{K}(\omega_c)$, $\Gamma_0 \propto \tilde{J}(0)$ and $\Gamma_2 \propto \tilde{J}(2\omega_c)$. While all the terms $|n\rangle\langle m|$ are eigenstates of the dephasing part with eigenvalues $(m - n)^2$ the parametric drive part does not have a steady state and drives the system towards the infinite temperatures.

5 Comparison with stochastic evolutions

Numerical algorithm for solving stochastic evolution equations have two advantages: They can work also beyond weak coupling and evolve states rather than density matrices. They also have several disadvantage: They tend to be slow because of the necessity to accumulating enough statistics; They are prone to long time drifts, and can be adversely affected by a poor random number generator and finally are prone to bugs. Our results on the Lindblad evolution can be used to test numerical algorithms for stochastic evolutions in those cases that both apply.

A comparison between Lindbladian evolutions of sections 4.1, 4.2, 4.3 and stochastic evolutions with Ornstein-Uhlenbeck process is shown in Fig. 7. Three cases have been studied: no control, control by constant H_0 and Bang-Bang. The weak coupling parameter is $\varepsilon = 0.15$ and the agreement is satisfactory. The numerical code is available upon request.

6 Summary

We derived the Lindbladian for controlled weakly stochastic evolutions both for small but finite ε and in the limit $\varepsilon \rightarrow 0$ for stationary control. Our results can be used to measure the power spectrum of the noise and to test numerical algorithms for solving stochastic evolution. Several examples are studied in detail.

Acknowledgment

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A Weak coupling expansion

The purpose of this appendix is to justify the estimate of Eq. (3.1). This requires a comparison of two different time orderings of the same exponent. Let us first ignore the

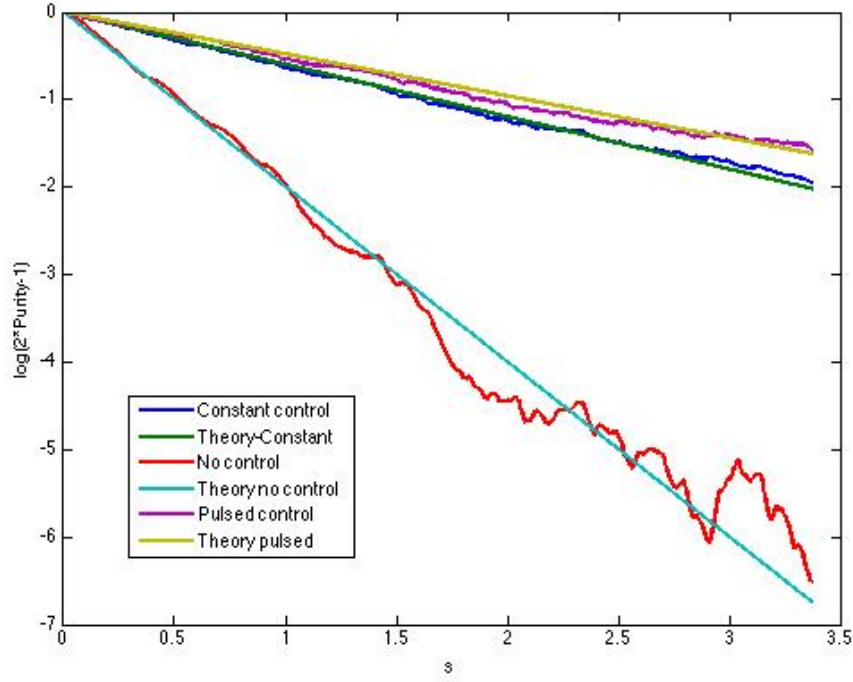


Figure 7: The logarithmic purity of the state, $\log \text{Tr}(\rho(s)^2)$ as a function of the coarse grained time s , Eq. (1.6) for stochastic evolutions and the corresponding Lindbladians. The weak coupling, Eq. (1.5), is $\varepsilon = 0.15$. The numerical grid $\Delta = 1$ and correlation time $\tau = 20\Delta$. The control parameter is $\omega_c \tau = 0.5\pi$, and the stochastic averaging is done on an ensemble of 500 runs.

ordering and consider

$$e^x = \sum \frac{x^n}{n!}, \quad x = - \int_0^t ds \mathcal{G}(s) \quad (\text{A.1})$$

The Taylor series for the exponent e^x is dominated by terms of order $n = O(x)$. In our case this gives $n \sim x \sim H^2 J \tau t \sim \varepsilon^2 t / \tau$. Writing the n -th term in the expansion

$$\frac{x^n}{n!} = \int_{0 \leq s_1 \leq \dots \leq s_n \leq t} \prod_{i=1}^n \mathcal{G}(s_i) ds_i \quad (\text{A.2})$$

we conclude that typically $s_{i+1} - s_i = O(t/n) = O(\tau/\varepsilon^2) \gg \tau$.

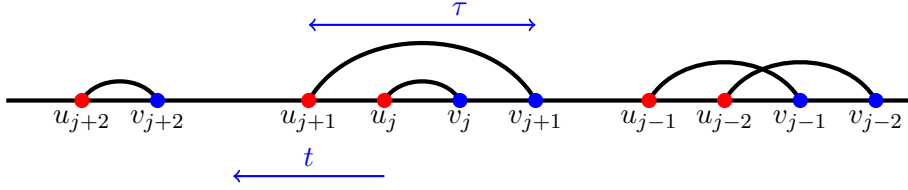


Figure 8: (u_j, v_j) make a dimer. The arc connecting the pair represents the short range interaction in Eq. (A.3). In a typical configuration the distance between dimers is large $O(\tau/\varepsilon^2)$ and there is at most one dimer in an interval of size τ . The dimer on the left is typical. The dimers on the right lead to the error. The approximation disregards the rare events.

Next let us consider the possible time orderings. The naive \tilde{T} time ordering with respect to the argument of $\mathcal{G}(s)$ implied by Eq. (A.2) differs from the correct T -ordering because the relation between \mathcal{G} and \mathcal{H}

$$\mathcal{G}(s_i) = -\frac{1}{2} \int_{u_i > v_i} du_i dv_i j(u_i - s_i) j(v_i - s_i) (\mathcal{H}(u_i) \mathcal{H}(v_i)) \quad (\text{A.3})$$

is non-local in time: The ordering of s_i does not guarantee the ordering of (u, v) . The fact that $j(u - s)$ is fast decaying implies, however, that the nonlocality in time is rather small $\sim \tau$. When $s_{i+1} - s_i \gg \tau$ the wrong ordering is almost the same as the correct one.

In order to estimate the error generated by using the \tilde{T} ordering consider more closely the two orderings. Each contribution to the exponent is given as in Eq. (A.2) by some choice of $0 \leq s_1 \leq \dots \leq s_n \leq t$ and we associate a choice of $u_i > v_i$ to each s_i as in Eq. (A.3). Typically $s_{i+1} - s_i \gg \tau \sim |u_i - s_i|, |v_{i+1} - s_{i+1}|$ and hence $v_{i+1} > u_i > v_i$ which implies that the two ordering are equivalent. If however there exists some i for which $u_i > v_{i+1}$ then the two expressions do not coincide.

Consider for example the case where $u_{j+1} > u_j > v_{j+1} > v_j$ while all other points

are at typical positions. This will lead to an error term of the type

$$\mathcal{U}_{n,\dots,j+2} \times \int_{s_{j+1} > s_j} ds_j ds_{j+1} \int_{u_{j+1} > u_j > v_{j+1} > v_j} du_j du_{j+1} dv_j dv_{j+1} \mathcal{H}(u_{j+1}) [\mathcal{H}(u_j), \mathcal{H}(v_{j+1})] \mathcal{H}(v_j) \quad (\text{A.4})$$

$$\times j(s_j - u_j) j(s_j - v_j) j(s_{j+1} - u_{j+1}) j(s_{j+1} - v_{j+1}) \times \mathcal{U}_{j-1,\dots,1} \quad (\text{A.5})$$

Here $\mathcal{U}_{n,\dots,j+2} = \int_{u_{j+1} \leq s_{j+2} \leq \dots \leq s_n \leq t} \prod_{i=j+2}^n \mathcal{G}(s_i) ds_i$ $\mathcal{U}_{j-1,\dots,1} = \int_{0 \leq s_1 \leq \dots \leq s_j \leq v_j} \prod_{i=1}^{j-1} \mathcal{G}(s_i) ds_i$ correspond to the (unitary) evolution before $t = v_j$ and after $t = u_{j+1}$. The integrand in Eq(A.4) is clearly fast decaying whenever its six integration variables are at inter-distance large compared to τ . It thus follows that the main contribution to the integral comes from a region of volume $\tau^5 t$. The integral is thus at most¹² of order of $\tau^5 t j^4 \|H\|^4 \sim \tau^3 t J^2 \|H\|^4 = \varepsilon^4 t / \tau$. Other nontypical cases (e.g. $u_j > u_{j+1} > v_{j+1} > v_j$) lead to error terms of a similar general form which again scale as $\varepsilon^4 t / \tau$.

The error terms we found are of the form $\int_0^t ds \mathcal{U}(s, t) \Delta \mathcal{G}(s) \mathcal{U}(0, s)$ for some $\Delta \mathcal{G}$ which is quartic in \mathcal{H} . This suggests defining an improved generator as $\mathcal{G} \mapsto \mathcal{G} + \Delta \mathcal{G}$. We however did not pursue this direction here.

B The spectrum of the super-operators of angular momenta

The adjoint representation $ad(S)$ of a representation S is constructed as the tensor product of S with its dual (contragredient) representation S^* . Since $SU(2)$ has a single representation in each dimension, it is obvious that $S^* \simeq S$. It thus follows that

$$ad(S) = S \otimes S^* = S \otimes S = 0 \oplus 1 \oplus 2 \oplus \dots \oplus (2S)$$

The spectrum (including multiplicities) of various operators such as $ad(S_z)$ and $\sum ad(S_j) ad(S_j)$ is then easily deduced

$$Spect(ad(S_z)) = \bigcup_{j=0,\dots,2S} \{m | m = -j, \dots, j\}$$

$$Spect\left(\sum_{i=x,y,z} (ad(S_i)^2)\right) = \bigcup_{j=0,\dots,2S} \{j(j+1) | m = -j, \dots, j\}$$

$$Spect\left(\sum_{i=x,y} (ad(S_i)^2)\right) = \bigcup_{j=0,\dots,2S} \{j(j+1) - m^2 | m = -j, \dots, j\}$$

In particular the eigenvalue zero appears in $Spect(ad(S_x)^2 + ad(S_y)^2)$ with trivial multiplicity 1. This last fact could also be deduced from Schur's lemma since by positivity $(ad(S_x)^2 + ad(S_y)^2)\rho = 0$ imply $ad(S_x)\rho = ad(S_y)\rho = 0$ and hence also $ad(S_z)\rho = -i[ad(S_x), ad(S_y)]\rho = 0$.

¹²If $H_I(t)$ changes slowly in time, then a tighter bound on the commutator is possible.

C Effective control

In dynamical decoupling one is interested in making \mathcal{L} small at the price of strong control, $\omega_c \tau \gg 1$. Since $\tilde{J}(\omega)$ is small for large argument and since the terms $\omega \neq 0$ in Eq. (1.10) tend to be of order $\tilde{J}(\omega_c)$ the “bad term” in \mathcal{L} is the one with $\omega = 0$. We say that the control is “effective” if $\tilde{H}_\alpha(\omega = 0) = 0$. The notion is independent of $J_\alpha(u)$, which is often not known.

Consider first strong continuous controls. Let $P_j(t)$ be the (instantaneous) spectral projections of H_c :

$$H_c(t) = \omega_c \sum e_j(t) P_j(t)$$

and suppose that $P_j(t)$ vary smoothly with t and that the $e_j(t)$ do not cross. Then, by the adiabatic theorem, for ω_c large

$$H_\alpha^I(t) \approx \sum_{j,k} e^{i\omega_c \int_0^t (e_j(u) - e_k(u)) du} P_j(t) H_\alpha P_k(t) \xrightarrow{\omega_c \rightarrow \infty} \sum_j P_j(t) H_\alpha P_j(t)$$

(in the sense of distributions.) It follows that the control is effective if, for all t ,

$$\sum_j P_j(t) H_\alpha P_j(t) = 0 \tag{C.1}$$

Bang-Bang at times t_j is effective if $\tilde{H}_\alpha(\omega = 0) = 0$, which is the case if $H_I(t)$ has zero average, i.e.

$$\forall \alpha, \quad \sum_j (t_{j+1} - t_j) V(t_j) H_\alpha V(t_j) = 0$$

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